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THE ASYMPTOTIC NUMBERS OF REGULAR TOURNAMENTS, EULERIAN DIGRAPHS AND EULERIAN ORIENTED GRAPHS

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Let $RT(n)$, $ED(n)$ and $EOG(n)$ be the number of labelled regular tournaments, labelled loop-free simple Eulerian digraphs, and labelled Eulerian oriented simple graphs, respectively, on n vertices. Then, as $n \to \infty$,

$$RT(n) = \left(\frac{2^{n+1}}{\pi n}\right)^{(n-1)/2} n^{1/2} e^{-1/2}\left(1 + O(n^{-1/2+\varepsilon})\right) \quad (n \text{ odd}),$$

$$ED(n) = \left(\frac{4^n}{\pi n}\right)^{(n-1)/2} n^{1/2} e^{-1/4}\left(1 + O(n^{-1/2+\varepsilon})\right), \text{ and}$$

$$EOG(n) = \left(\frac{3^{n+1}}{4\pi n}\right)^{(n-1)/2} n^{1/2} e^{-3/8}\left(1 + O(n^{-1/2+\varepsilon})\right),$$

for any $\varepsilon > 0$. The last two families of graphs are also enumerated by their numbers of edges. The proofs use the saddle point method applied to appropriate n-dimensional integrals.

1. Introduction

A *tournament* is a digraph in which, for each pair of distinct vertices v and w, either (v, w) or (w, v) is an edge, but not both. A tournament is *regular* if the in-degree is equal to the out-degree at each vertex. Let $RT(n)$ be the number of labelled regular tournaments with n vertices. It is easy to see that $RT(n) = 0$ if n is even.

By an *eulerian* digraph we mean a digraph in which the in-degree is equal to the out-degree at each vertex. (Thus, the regular tournaments are exactly the eulerian tournaments.) Let $ED(n)$ be the number of labelled loop-free simple eulerian digraphs with n vertices. Allowing simple loops would multiply $ED(n)$ by exactly 2^n, since loops do not affect the eulerian property. Let $EOG(n)$ be the number of labelled loop-free simple eulerian digraphs in which at most one of the edges (v, w) and (w, v) are permitted for any distinct v and w.

In this paper we determine the asymptotic values of $RT(n)$, $ED(n)$ and $EOG(n)$, and the last two classes by their numbers of edges (within limits). The method in each case will be the same: we identify the required quantity as a coefficient in an n-variable power series, and estimate it by applying the saddle-point method to the integral provided by Cauchy's Theorem. Since the parameter which is tending to ∞ is the number of dimensions, the application of the saddle-point method has an analytic flavour different from that of most fixed-dimensional problems. In particular, the choice of contour is trivial but substantial work is required to demonstrate that

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the parts of the contour where the integrand is small contribute negligibly to the result. For another calculation similar to those here, see McKay and Wormald [4].

Exact values of $RT(n)$ for $n \le 21$, $ED(n)$ for $n \le 16$ and $EOG(n)$ for $n \le 15$ can be found in [2]. They are in excellent agreement with our estimates, if we note that the values of $ED(n)$ given in [2] do not permit loops (contrary to the claim made there). Exact formulas for $RT(n)$ and $ED(n)$ can be found in [1]. However, they involve multiple summations over roots of unity and do not seem suitable for asymptotic analysis.

The only previous directly related results that we are aware of are due to Joel Spencer [5]. In particular, Spencer evaluates $RT(n)$ to within a factor of $(1+o(1))^n$.

2. An integral

In this section we will evaluate an n-dimensional integral which occurs in each of the estimations we wish to perform. We will need the following lemma, which is well known.

Lemma 2.1. *The surface area of the n-dimensional sphere of radius ρ is*

$$2\pi^{n/2}\rho^{n-1}/\Gamma(n/2). \qquad \blacksquare$$

For $t \ge 0$ and $n \ge 1$, define $U_n(t) = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n \mid |x_i| \le t \text{ for } 1 \le i \le n\}$.

Theorem 2.1. *Let a, b and c be real numbers with $a > 0$. Let $0 < \varepsilon < 1/8$, and let $n \ge 2$ be an integer. Define*

$$J = J(a,b,n) =$$
$$\int \exp\biggl(-a \sum_{1\le j<k\le n} (\theta_j-\theta_k)^2 + b \sum_{1\le j<k\le n} (\theta_j-\theta_k)^4 + \frac{c}{n^2}\Bigl(\sum_{1\le j<k\le n} (\theta_j-\theta_k)^2\Bigr)^2\biggr)\, d\boldsymbol{\theta}',$$

where the integral is over $\boldsymbol{\theta}' = (\theta_1, \theta_2, \ldots, \theta_{n-1}) \in U_{n-1}(n^{-1/2+\varepsilon})$ with $\theta_n = 0$. Then, as $n \to \infty$,

$$J = n^{1/2}\Bigl(\frac{\pi}{an}\Bigr)^{(n-1)/2} \exp\Bigl(\frac{6b+c}{4a^2} + O(n^{-1/2+4\varepsilon})\Bigr).$$

Proof. Define $V = U_{n-1}(n^{-1/2+\varepsilon})$. We begin by approximately diagonalising the integrand. Let $T : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ be the linear transformation defined by $T : \boldsymbol{\theta}' \mapsto \boldsymbol{y} = (y_1, y_2, \ldots, y_{n-1})$, where

$$y_j = \theta_j - \sum_{k=1}^{n-1} \theta_k/(n+n^{1/2})$$

for $1 \le j \le n-1$. Let $V_1 = T(V)$. For $k \ge 1$ define $\mu_k = \mu_k(\boldsymbol{y}) = \sum_{j=1}^{n-1} y_j^k$. Items (1)–(5) can be verified by straightforward calculations:

$$V_1 = \{\, \boldsymbol{y} \mid |y_j + \mu_1/(n^{1/2}+1)| \le n^{-1/2+\varepsilon} \text{ for } 1 \le j \le n-1 \,\}. \tag{1}$$

(2) $$\mu_1 = n^{-1/2} \sum_{j=1}^{n-1} \theta_j.$$

(3) $$\sum_{1\le j<k\le n} (\theta_j - \theta_k)^2 = n\mu_2.$$

(4) $$\sum_{1\le j<k\le n} (\theta_j - \theta_k)^4 = n\mu_4 + 3\mu_2^2 - \frac{4n^{1/2}}{n^{1/2}+1}\mu_1\mu_3 + \frac{6}{(n^{1/2}+1)^2}\mu_1^2\mu_2 + \frac{n^{1/2}+3}{(n^{1/2}+1)^3}\mu_1^4.$$

(5) $$\det(T) = n^{1/2}.$$

If $\boldsymbol{y} \in V_1$, it follows from (2) that $|\mu_1| \le n^{\varepsilon}$ and from (1) that $V_1 \subseteq U_{n-1}(2n^{-1/2+\varepsilon})$. The latter implies that $\mu_2 \le 4n^{2\varepsilon}$ and $\mu_3 \le 8n^{-1/2+3\varepsilon}$.

Since the integrand of J is real and positive, we conclude from (3), (4) and (5) that $J = n^{1/2}\exp(O(n^{-1/2+4\varepsilon}))J_1$, where

$$J_1 = \int_{V_1} F(\boldsymbol{y})\, d\boldsymbol{y}, \quad F(\boldsymbol{y}) = \exp(-an\mu_2 + bn\mu_4 + 3b\mu_2^2 + c\mu_2^2).$$

Since $V_1 \subseteq U_{n-1}(2n^{-1/2+\varepsilon})$ we have that $\mu_4 \le 4n^{-1+2\varepsilon}\mu_2$. Thus

(6) $$F(\boldsymbol{y}) = \exp(-an\mu_2(1 + O(n^{-1+2\varepsilon}))).$$

For $\rho \ge 0$, define $S'_\rho = S_\rho \cap V_1$, where $S_\rho = \{\boldsymbol{y} \mid \mu_2 = \rho^2\}$. The volume of S'_ρ is $O(1)(2\pi e/n)^{(n-1)/2}\rho^{n-2}$ by Lemma 2.1, and is zero if $\rho > 2n^{\varepsilon}$. By (6),

(7) $$\int_{S'_\rho} F(\boldsymbol{y})\, d\boldsymbol{y} = O(1)\Big(\frac{2\pi e}{n}\Big)^{(n-1)/2} \rho^{n-2}\exp(-an\rho^2(1 + O(n^{-1+2\varepsilon}))).$$

The function $g(\rho) = \rho^{n-2}\exp(-an\rho^2(1 + O(n^{-1+2\varepsilon})))$ has its maximum near $\rho = (2a)^{-1/2}$; if $|\rho - (2a)^{-1/2}| > n^{-1/2+2\varepsilon}$ and $\rho \le 2n^{\varepsilon}$, then

$$g(\rho) \le (2ae)^{-n/2}\exp(-c_1 n^{4\varepsilon})$$

for some constant $c_1 > 0$. Consequently, defining

$$W = U_{n-1}(2n^{-1/2+\varepsilon}) \cap \{\, \boldsymbol{y} \mid |\mu_2^{1/2} - (2a)^{-1/2}| \le n^{-1/2+2\varepsilon} \,\},$$

we have $J_1 = J_2 + \Delta$, where

$$J_2 = \int_{V_1\cap W} F(\boldsymbol{y})\, d\boldsymbol{y}, \quad\text{and}\quad |\Delta| \le O(n)\Big(\frac{\pi}{an}\Big)^{n/2}\exp(-c_1 n^{4\varepsilon}).$$

Since $\mu_2^2 = (2a)^{-2}(1 + O(n^{-1/2+2\varepsilon}))$ if $\boldsymbol{y} \in W$,

(8) $$J_2 = \exp\Big(\frac{3b+c}{4a^2} + O(n^{-1/2+2\varepsilon})\Big)J_3, \text{ where}$$

$$\begin{aligned} J_3 &= \int_{V_1\cap W} \exp(-an\mu_2 + bn\mu_4)\, d\boldsymbol{y} \\ &= \int_W \exp(-an\mu_2 + bn\mu_4)\, d\boldsymbol{y} - \int_{W\setminus V_1} \exp(-an\mu_2 + bn\mu_4)\, d\boldsymbol{y}. \end{aligned}$$

Let J_4 and Δ' denote the two integrals in (8), respectively. If $V(\rho)$ denotes the volume of $(W \setminus V_1) \cap S_\rho$, then clearly

$$\Delta' \leq \int_{|\rho-(2a)^{-1/2}|\leq n^{-1/2+2\varepsilon}} V(\rho)\exp(-an\rho^2(1+O(n^{-1+2\varepsilon})))\,d\rho. \tag{9}$$

We will bound $V(\rho)$ with the help of a statistical argument. Let $Y_1, Y_2, \ldots, Y_{n-1}$ be independent random variables with the normal density $N(0, \rho^2/n)$. Then

$$\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_{n-1}) = \frac{\rho\,(Y_1, Y_2, \ldots, Y_{n-1})}{(Y_1^2 + Y_2^2 + \cdots + Y_{n-1}^2)^{1/2}}$$

is a random point on S_ρ. From the weights of the tails of the normal and χ^2 distributions we find that the events $|Y_j| \leq n^{-1/2+\varepsilon}/2$ $(1 \leq j \leq n-1)$, $|\sum Y_j| \leq n^\varepsilon/4$ and $\sum Y_j^2 < \rho^2/2$ occur simultaneously with probability at least $1-\exp(-c_2 n^{2\varepsilon})$ for some constant $c_2 > 0$. However, these conditions together imply that $\boldsymbol{Z} \in V_1$, by (1). We conclude that $V(\rho)$ is at most $\exp(-c_2 n^{2\varepsilon})$ times the volume of S_ρ. Applying this to (9), we obtain

$$\Delta' \leq O(1)\Big(\frac{\pi}{an}\Big)^{n/2}\exp(-c_2 n^{2\varepsilon}).$$

Finally, consider

$$J_4 = \int_{U_{n-1}(2n^{-1/2+\varepsilon})} \exp(-an\mu_2 + bn\mu_4)\,d\boldsymbol{y}.$$

By the same argument as before, $J_4 = J_3 + \Delta''$, where

$$\Delta'' \leq O(1)\Big(\frac{\pi}{an}\Big)^{n/2}\exp(-c_3 n^{2\varepsilon})$$

for some constant $c_3 > 0$ and

$$\begin{aligned}
J_4 &= \Big(\int_{-2n^{-1/2+\varepsilon}}^{2n^{-1/2+\varepsilon}} \exp(-anx^2 + bnx^4)\,dx\Big)^{n-1} \\
&= \Big(\int_{-2n^{-1/2+\varepsilon}}^{2n^{-1/2+\varepsilon}} \exp(-anx^2)(1 + bnx^4 + O(n^2x^8))\,dx\Big)^{n-1} \\
&= \Big(\frac{\pi}{an}\Big)^{(n-1)/2}\Big(1 + \frac{3b}{4a^2n} + O(n^{-2})\Big)^{n-1} \\
&= \Big(\frac{\pi}{an}\Big)^{(n-1)/2}\exp\Big(\frac{3b}{4a^2} + O(n^{-1})\Big).
\end{aligned}$$

Combining our estimates now leads easily to the theorem. ∎

3. Derivation of the principal theorems

We begin with a technical lemma whose proof is too elementary to include.

Lemma 3.1.
(a) *For* $|x| \leq \pi/2$, $\quad \cos(x) \leq \exp(-x^2/2)$.
(b) *For* $0 \leq \lambda \leq 1$ *and any real* x, $|1-\lambda+\lambda\cos(x)| \leq \exp(-\lambda x^2/2 + \lambda x^4/24)$. ∎

We now have the necessary apparatus to perform the estimations we have promised. In the case of $RT(n)$ we will give the proof in detail, but in the other cases we will be content with an outline.

Theorem 3.1. *As* $n \to \infty$ *with* n *odd,*

$$RT(n) = \frac{2^{(n^2-1)/2}e^{-1/2}}{\pi^{(n-1)/2}n^{n/2-1}}(1+O(n^{-1/2+\varepsilon}))$$

for any $\varepsilon > 0$.

Proof. Without loss of generality, take $\varepsilon < 1/2$. The generating function

$$\prod_{1\leq j<k\leq n}(x_j^{-1}x_k + x_jx_k^{-1})$$

enumerates tournaments by the excess of out-degree over in-degree at each vertex. Thus, $RT(n)$ is the constant term. By Cauchy's Theorem,

$$RT(n) = \frac{1}{(2\pi i)^n}\oint\cdots\oint \frac{\prod_{1\leq j<k\leq n}(x_j^{-1}x_k + x_jx_k^{-1})}{x_1x_2\cdots x_n}\,dx_1dx_2\ldots dx_n,$$

where each integration is around a simple closed contour encircling the origin once in the anticlockwise direction. Substituting $x_j = e^{i\theta_j}$ $(1 \leq j \leq n)$, we obtain

$$(10) \qquad RT(n) = \frac{2^{n(n-1)/2}}{(2\pi)^n}I, \quad I = \int_{U_n(\pi)}\prod_{1\leq j<k\leq n}\cos(\theta_j-\theta_k)\,d\boldsymbol{\theta},$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$. Due to the periodic nature of the integrand, we will treat each θ_j as having values mod 2π. Note that the integrand in (10) always lies in the real interval $[-1,1]$. Also, translation of any θ_j by π leaves the integrand unchanged (since n is odd).

We will begin the evaluation of I with the part of the integrand which will turn out to give the major contribution. Let I_1 be the contribution to I of those $\boldsymbol{\theta}$ such that either $|\theta_j - \theta_n| \leq n^{-1/2+\varepsilon/4}$ or $|\theta_j - \theta_n + \pi| \leq n^{-1/2+\varepsilon/4}$ for $1 \leq j \leq n$, where θ_j values are taken mod 2π as we stated earlier. The contributions to I_1 with different values of θ_n are clearly the same. Also, both the integrand and the region of integration are invariant under translation of any θ_j by π. Thus

$$I_1 = 2^n\pi\int_{U_{n-1}(n^{-1/2+\varepsilon/4})}\prod_{1\leq j<k\leq n}\cos(\theta_j-\theta_k)\,d\boldsymbol{\theta}',$$

where the integration is with respect to $\boldsymbol{\theta}' = (\theta_1, \theta_2, \ldots, \theta_{n-1})$ with $\theta_n = 0$. Now we can expand

$$\prod_{1\le j<k\le n} \cos(\theta_j - \theta_k) = \exp\Big(\sum_{1\le j<k\le n} \log\cos(\theta_j - \theta_k)\Big) =$$

$$\exp\Big(-\frac{1}{2}\sum_{1\le j<k\le n}(\theta_j - \theta_k)^2 - \frac{1}{12}\sum_{1\le j<k\le n}(\theta_j - \theta_k)^4 + O(n^{-1+3\varepsilon/2})\Big).$$

Thus, by Theorem 2.1,

$$I_1 = 2^n\pi\Big(\frac{2\pi}{n}\Big)^{(n-1)/2} n^{1/2}e^{-1/2}(1 + O(n^{-1/2+\varepsilon})). \tag{11}$$

From (10) and (11) we see that I_1 matches our claimed value for $RT(n)$, so we must now show that the other parts of the integral I are negligible.

For $0 \le j \le 31$, define the interval $A_j = [(j-1)\pi/16, j\pi/16]$. For any $\boldsymbol{\theta} \in U_n(\pi)$, at least one of the 16 intervals $A_0 \cup A_1, A_2 \cup A_3, \ldots, A_{30} \cup A_{31}$ contains $n/16$ or more of the θ_j. Let us suppose that this is true of $A_0 \cup A_1$ (thereby undercounting the possibilities by at most a factor of 16). Define $B = A_3 \cup \cdots \cup A_{14} \cup A_{19} \cup \cdots \cup A_{30}$. If $\theta_j \in B$ and $\theta_k \in A_0 \cup A_1$, then $|\cos(\theta_j - \theta_k)| \le \cos(\pi/16)$. From this it easily follows that, for sufficiently large n, the contribution to I of all the cases where n^ε or more of the θ_j lie in B is at most $\exp(-c_1 n^{1+\varepsilon})I_1$ for some $c_1 > 0$. Thus, with an undercount of at most 16, we can suppose that at least $n - n^\varepsilon$ of the θ_j lie in $A_{31} \cup A_0 \cup A_1 \cup A_2 \cup A_{15} \cup \cdots \cup A_{18}$. At the expense of another factor of 2^n, we can suppose that $|\theta_j| \le \pi/2$ for all j and that $|\theta_j| \le \pi/8$ for at least $n - n^\varepsilon$ of the θ_j. Now define $I_2(r)$ to be the contribution to I of those $\boldsymbol{\theta}$ such that

(i) $3\pi/16 \le |\theta_j| \le \pi/2$ for r values of j,
(ii) $|\theta_j| \le \pi/8$ for at least $n - n^\varepsilon$ values of j, and
(iii) $\pi/8 \le |\theta_j| \le 3\pi/16$ for any other values of j.

Clearly $I_2(r) = 0$ if $r > n^\varepsilon$. If θ_j and θ_k are in classes (i) and (ii), respectively, then $|\cos(\theta_j - \theta_k)| \le \cos(\pi/16)$, while if they are both in classes (ii) or (iii), $|\cos(\theta_j - \theta_k)| \le \exp(-(\theta_j - \theta_k)^2/2)$, by Lemma 3.1(a). Using $|\cos(\theta_j - \theta_k)| \le 1$ for the other cases, we find

$$|I_2(r)| \le \pi^r\binom{n}{r}\cos(\pi/16)^{r(n-n^\varepsilon)}|I_2'(n-r)|, \tag{12}$$

where

$$I_2'(m) = \int_{U_m(3\pi/16)} \exp\Big(-\frac{1}{2}\sum_{1\le j<k\le m}(\theta_j - \theta_k)^2\Big)\,d\theta_1\cdots d\theta_m.$$

Since θ_n ranges over $[-3\pi/16, 3\pi/16]$ and the integrand is everywhere positive, we can apply the transformation T of Theorem 2.1 (using m in place of n) to easily obtain

$$I_2'(m) \le \frac{3}{8}\pi m^{1/2}\Big(\frac{2\pi}{m}\Big)^{(m-1)/2}.$$

Substituting back into (12) we find that

$$2^n \sum_{r=1}^{n^\varepsilon} |I_2(r)| \le |I_1| \exp(-c_2 n + o(n))$$

for some $c_2 > 0$. We conclude that the only substantial contribution must come from the case $r = 0$.

Next, define $I_3(h)$ be the contribution to I of those $\boldsymbol{\theta}$ such that

(i) $|\theta_n| \le 3\pi/16$,

(ii) $n^{-1/2+\varepsilon/4} \le |\theta_j - \theta_n| \le 3\pi/8$ for h values of j, and

(iii) $|\theta_j - \theta_n| \le n^{-1/2+\varepsilon/4}$ for the remaining values of j.

Clearly $|I_3(h)| \le 3\pi|I_3'(h)|/16$, where $|I_3'(h)|$ is the same integral over $\boldsymbol{\theta}'$ with $\theta_n = 0$. Now apply the bound $\cos(\theta_j - \theta_k) \le \exp(-(\theta_j - \theta_k)^2/2)$ and transform the $\boldsymbol{\theta}'$ to $\boldsymbol{y}$ using the transformation T of Theorem 2.1. The values of $\boldsymbol{\theta}'$ contributing to $I_3'(h)$ for $h \ge 1$ map to a subset of those $\boldsymbol{y}$ such that either $|\mu_1| > n^{\varepsilon/4}/2$ or $|y_j| > n^{-1/2+\varepsilon/4}/2$ for some j. Since the contribution to

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2} n\mu_2)\, d\boldsymbol{y}$$

of those $\boldsymbol{y}$ is $O(n)(2\pi/n)^{(n-1)/2} \exp(-c_3 n^{\varepsilon/2})$ for some $c_3 > 0$, we conclude that

$$2^n \sum_{h=1}^{n-1} |I_3(h)| \le O(n) \exp(-c_3 n^{\varepsilon/2}) |I_1|.$$

The remaining case, $h = 0$, is covered by I_1. The theorem follows. ∎

In the case of $ED(n)$ and $EOG(n)$, we will omit the fine detail. The missing parts of the calculations are essentially the same as the corresponding parts of Theorem 2.1.

Theorem 3.2. *As $n \to \infty$,*

$$ED(n) = \frac{2^{n^2-n} e^{-1/4}}{\pi^{(n-1)/2} n^{n/2-1}} (1 + O(n^{-1/2+\varepsilon})) \text{ and}$$

$$EOG(n) = \frac{3^{(n^2-1)/2} e^{-3/8}}{2^{n-1} \pi^{(n-1)/2} n^{n/2-1}} (1 + O(n^{-1/2+\varepsilon}))$$

for any $\varepsilon > 0$.

Proof. $ED(n)$ is the constant term in $\prod_{1 \le j < k \le n} (1 + x_j^{-1} x_k)(1 + x_j x_k^{-1})$. Applying Cauchy's Theorem as in Theorem 2.1, we find

$$ED(n) = \frac{2^{n^2-n}}{(2\pi)^n} \int_{U_n(\pi)} \prod_{1 \le j < k \le n} \left(\frac{1}{2} + \frac{1}{2} \cos(\theta_j - \theta_k) \right) d\boldsymbol{\theta}.$$

Arguments similar to those of Theorem 2.1 show that the dominant contribution to the integral comes when the θ_j are clustered together. (The only substantial differences are that we don't have the invariance under translation by π and that we require Lemma 3.1(b) in place of Lemma 3.1(a).) If the θ_j are clustered together, we can expand

$$\prod_{1\le j<k\le n}\left(\frac{1}{2}+\frac{1}{2}\cos(\theta_j-\theta_k)\right)=\exp\left(-\frac{1}{4}\sum_{1\le j<k\le n}(\theta_j-\theta_k)^2-\frac{1}{96}\sum_{1\le j<k\le n}(\theta_j-\theta_k)^4-\cdots\right)$$

and apply Theorem 2.1.

Similarly, $EOG(n)$ is the constant term in $\prod_{1\le j<k\le n}(1+x_j^{-1}x_k+x_jx_k^{-1})$. Application of Cauchy's Theorem gives

$$EOG(n)=\frac{3^{n(n-1)/2}}{(2\pi)^n}\int_{U_n(\pi)}\prod_{1\le j<k\le n}\left(\frac{1}{3}+\frac{2}{3}\cos(\theta_j-\theta_k)\right)d\boldsymbol{\theta},$$

and the same approach yields the desired result. ∎

We turn now to the enumeration of eulerian digraphs by their numbers of edges. Define $EOG(n,m)$ to be the number of labelled loop-free simple eulerian digraphs with n vertices and m edges, and $ED(n,m)$ to be the number of labelled loop-free simple eulerian digraphs with n vertices and m edges. We will compute asymptotic expressions for $EOG(n,m)$ and $ED(n,m)$ via a common generalisation. Let us call (v,w) a *type-1* edge if (w,v) is not also present, and a *type-2* edge if (w,v) is present. Thus, the pair $\{(v,w),(w,v)\}$, if both present, comprises two type-2 edges. Now define $F(n,m_1,m_2)$ to be the number of labelled loop-free simple eulerian digraphs with exactly m_1 type-1 edges and $2m_2$ type-2 edges. Clearly $EOG(n,m)=F(n,m,0)$ and $ED(n,m)=\sum_{m_1+2m_2=m}F(n,m_1,m_2)$. Write $N=\binom{n}{2}$.

Theorem 3.3. *Let $0<c_1<c_2<1$ and $\varepsilon>0$. Then, as $n\to\infty$,*

$$F(n,\beta N,m_2)=\frac{2^{\beta N-(n-1)/2}e^{-1/2}}{\pi^{n/2}n^{n/2}\beta^{\beta N+n/2}(1-\beta)^{(1-\beta)N+1/2}}\binom{(1-\beta)N}{m_2}(1+O(n^{-1/2+\varepsilon}))$$

uniformly for $c_1\le\beta\le c_2$ and $0\le m_2\le(1-\beta)N$, provided βN is an integer.

Proof. Irrespective of which βN type-1 edges are present, there are exactly $\binom{(1-\beta)N}{m_2}$ choices for the type-2 edges. Hence it will suffice to treat the case $m_2=0$. $F(n,\beta N,0)$ is the coefficient of $t^{\beta N}x_1^0\cdots x_n^0$ in

$$\Phi(t,\boldsymbol{x})=\prod_{1\le j<k\le n}(1+tx_j^{-1}x_k+tx_jx_k^{-1}).$$

We will extract this coefficient by Cauchy's Theorem, as in the previous theorems, integrating each x_j around the unit circle and t around the circle of radius R, where $R=\beta/(2(1-\beta))$. Change variables to $(\phi,\theta_1,\theta_2,\ldots,\theta_n)$ by $x_j=e^{i\theta_j}$ and $t=Re^{i\phi}$. By methods basically the same as those used in Theorem 3.1, we find that the integral is dominated by the contributions where $|\theta_j-\theta_k|\le n^{-1/2+\varepsilon/4}$ for $1\le j<k\le n$

and $|\phi| \le n^{-1+\varepsilon/2}$. We will omit the proof of this fact and proceed to the estimation of the dominant part.

We begin by integrating with respect to t. Let $\Theta(\boldsymbol{\theta})$ be the coefficient of $t^{\beta N}$ in $\Phi(t,\boldsymbol{x})$. Then

$$\Theta(\boldsymbol{\theta}) = \frac{\prod_{1\le j<k\le n}(1+2R\cos(\theta_j-\theta_k))}{2\pi R^{\beta N}} \int_{-\pi}^{\pi} \Psi(\boldsymbol{\theta})\,d\phi,$$

where

$$\Psi(\boldsymbol{\theta}) = e^{-i\beta N\phi} \prod_{1\le j<k\le n} (1+\lambda_{jk}(e^{i\phi}-1)), \quad \lambda_{jk} = \frac{2R\cos(\theta_j-\theta_k)}{1+2R\cos(\theta_j-\theta_k)}.$$

If $|\phi| \le n^{-1+\varepsilon/2}$ and $|\theta_j-\theta_k| \le n^{-1/2+\varepsilon/4}$ for $1\le j<k\le n$, we have

$$\log\Psi(\boldsymbol{\theta}) = -\beta iN\phi + \sum_{1\le j<k\le n}\Big(\lambda_{jk}i\phi - \frac{1}{2}\lambda_{jk}(1-\lambda_{jk})\phi^2 + O(n^{-3+3\varepsilon/2})\Big),$$

$$\sum_{1\le j<k\le n}\lambda_{jk} = \beta N - \frac{1}{2}\beta(1-\beta)\sum_{1\le j<k\le n}(\theta_j-\theta_k)^2 + O(n^{\varepsilon}), \quad \text{and}$$

$$\sum_{1\le j<k\le n}\lambda_{jk}(1-\lambda_{jk}) = \beta(1-\beta)N + O(n^{1+\varepsilon}),$$

and so

$$\log\Psi(\boldsymbol{\theta}) = -\frac{1}{2}i\beta(1-\beta)\phi\sum_{1\le j<k\le n}(\theta_j-\theta_k)^2 - \beta(1-\beta)N\phi^2 + O(n^{-1+2\varepsilon}).$$

Consequently,

$$\int_{-\pi}^{\pi}\Psi(\boldsymbol{\theta})\,d\phi = \frac{2\pi^{1/2}}{n\beta^{1/2}(1-\beta)^{1/2}}\exp\Big(-\frac{\beta(1-\beta)}{8N}\Big(\sum_{1\le j<k\le n}(\theta_j-\theta_k)^2\Big)^2 + O(n^{-1+2\varepsilon})\Big),$$

which yields

$$\Theta(\boldsymbol{\theta}) = \frac{(1+2R)^N}{\pi^{1/2}n\beta^{1/2}(1-\beta)^{1/2}R^{\beta N}} \times \exp\Big(-\frac{\beta(1-\beta)}{4n^2}\Big(\sum_{1\le j<k\le n}(\theta_j-\theta_k)^2\Big)^2$$
$$+ \sum_{1\le j<k\le n}\log\Big(\frac{1+2R\cos(\theta_j-\theta_k)}{1+2R}\Big) + O(n^{-1+2\varepsilon})\Big).$$

We can now expand

$$\log\Big(\frac{1+2R\cos(\theta_j-\theta_k)}{1+2R}\Big) = -\frac{1}{2}\beta(\theta_j-\theta_k)^2 + \frac{1}{24}\beta(1-3\beta)(\theta_j-\theta_k)^4 + O(n^{-3+3\varepsilon/2})$$

and complete the proof with the help of Theorem 2.1. ∎

As earlier noted, $EOG(n,m) = F(n,m,0)$. For large fixed n, the maximum of $EOG(n,m)$ occurs at $m = n^2/3 - n/2 + o(n)$. In fact, for $|t| = o(n^{4/3})$,

$$EOG(n, n^2/3-n/2+t) = \frac{3^{(n^2+1)/2}}{2^{n-1/2}\pi^{n/2}n^{n/2}} \exp\Bigl(-\frac{3}{8}-\frac{9}{2}n^{-2}t^2+O(n^{-4}|t|^3+n^{-1/2+\varepsilon})\Bigr).$$

Summation over t recovers the formula for $EOG(n)$ in Theorem 3.2.

Theorem 3.4. *Let $0 < c_1 < c_2 < 2$ and $\varepsilon > 0$. Then, as $n \to \infty$,*

$$ED(n, \alpha N) = \frac{e^{-1/4}2^{n^2-n+1/2}}{\pi^{n/2}n^{n/2}(2-\alpha)^{n^2}} \Bigl(\frac{2-\alpha}{\alpha}\Bigr)^{(\alpha n-\alpha+1)n/2} (1 + O(n^{-1/2+\varepsilon}))$$

uniformly for $c_1 \le \alpha \le c_2$.

Proof. As previously stated, $ED(n, \alpha N) = \sum_{m_1+2m_2=\alpha N} F(n, m_1, m_2)$. From Theorem 3.3, we find that the bulk of the sum comes when m_1 is close to $m(\alpha)N$, where

$$m(\alpha) = \alpha - \alpha^2/2 - \alpha(2-\alpha)/(2n).$$

In fact, for small t,

$$\begin{aligned} &F(n, (m(\alpha)+t)N, (\alpha - m(\alpha) - t)N/2) \\ &\quad = \frac{e^{-1/2}2^{n^2-n+5/2}}{\pi^{n/2+1/2}n^{n/2+1}(2-\alpha)^{n^2+2}} \Bigl(\frac{2-\alpha}{\alpha}\Bigr)^{(\alpha n-\alpha+1)n/2} \\ &\quad\quad \times \exp\Bigl(-\frac{n^2}{\alpha^2(2-\alpha)^2}t^2 + O(n^2|t|^3 + n^{-1/2+\varepsilon})\Bigr). \end{aligned}$$

Summing this equation over those t for which the third argument is an integer, we obtain the required expression. ∎

The maximum of $ED(n, \alpha N)$ occurs when $\alpha = 1 + o(n^{-1})$. In fact, if $t = o(n^{3/2})$,

$$ED(n, N+t) = \frac{e^{-1/4}2^{n^2-n+1/2}}{\pi^{n/2}n^{n/2}} \exp(-2n^{-2}t^2 + O(n^{-3}t^2 + n^{-1/2+\varepsilon})).$$

Summing over t recovers our formula for $ED(n)$.

Theorem 3.1 can be used in conjunction with a theorem of Spencer [5] to derive the asymptotic number of tournaments with a given score sequence, provided the tournaments are not very far from being regular. With a non-trivial amount of extra work, the proof method of Theorem 3.1 can be used to widen that estimate [3]. Similarly, asymptotic enumeration of eulerian digraphs with a given degree sequence (not too far from regular) is definitely within reach of our methods. We hope to return to this question in a future paper.

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